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Three-Graviton Vertex Calculation and Divergence Analysis of Higher-Derivative Quantum Gravity

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Higher-derivative contributions to the three-graviton vertex are calculated for 4-derivative quantum gravity, and its divergence is analyzed. It is shown that new divergences exist in the theory.

As is well known, Einstein gravity is nonrenormalizable (Alvarez, 1989; Goroff and Sagnotti, 1986; van de Ven, 1992), as there is an infinite number of divergences in the theory. So it has been suggested to construct a renormalizable theory by allowing oneself the freedom to modify general relativity by, e.g., adding higher-derivative terms to the Lagrangian (Stelle, 1977). However, considering the contributions of the higher-derivative terms to three-graviton vertex, we find that the theory will include new divergences.

1. CONTRIBUTIONS OF HIGHER-DERIVATIVE TERMS TO THREE-GRAVITON VERTEX

In general, the action of higher-derivative gravity is (Stelle, 1977)

$$S_{\rm gr} = -\int d^4x \sqrt{-g}(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \kappa^{-2}\gamma R) \tag{1}$$

where $\gamma = 2$, $\kappa^2 = 32\pi G$, $R^{\lambda}_{\ \mu\alpha\nu} = \partial_\nu \Gamma^{\lambda}_{\mu\alpha} \ldots$. The gravitational field variables are defined by the contravariant metric density:

$$h^{\mu\nu} = \kappa^{-1}(\sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}) \tag{2}$$

where $\eta^{\mu\nu} = \text{diag}\,(-+++)$.

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The standard quantized program gives the BRS-invariant generating functional as follows:

$$Z[T_{\mu\nu}, \overline{\theta}_\alpha, \theta^\beta, P_{\mu\nu}, Q_\alpha] = N^{-1} \int [dh^{\mu\nu}][dC^\alpha][d\overline{C}_\beta]\exp(iS_{\text{eff}} + i \int d^4xL_s) \tag{3}$$

where

$$L_s = \kappa T_{\mu\nu}h^{\mu\nu} + \overline{\theta}_\alpha C^\alpha + \overline{C}_\alpha\theta^\alpha + \kappa P_{\mu\nu}D_\alpha^{\mu\nu}C^\alpha + \kappa^2 Q_\alpha\partial_\beta C^\alpha C^\beta \tag{4}$$

$$S_{\text{eff}} = S_{\text{gr}} - \frac{1}{2}\kappa^2\lambda^{-1} \int d^4x\, \partial^\mu h_{\mu\tau}\Box\partial_\nu h^{\nu\tau} + \int d^4x\overline{C}_\mu\partial_\nu D_\alpha^{\mu\nu}C^\alpha \tag{5}$$

and C, $\overline{C}$ are the Faddeev–Popov ghost and antighost, respectively.

From equations (4) and (5), one finds that the ghost propagator behaves like k^{-2}, the vertex of h-C-$\overline{C}$ like k^2 (Capper *et al.*, 1973), and both h-C-P and C-C-Q like k^1. The graviton propagator has contributions from terms h^2, particularly from higher-derivative terms, of the action, and its behavior is as k^{-4} at large momentum (Stelle, 1977).

The three-graviton vertex should have contributions from all h^3 terms in $S_{\text{gr}}(1)$ i.e., it must have contributions coming from both R^2 and R. We find that the vertex is

$$\Theta(k_1, k_2, k_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3} = \Theta^{(2)}(k_1, k_2, k_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3} + \Theta^{(4)}(k_1, k_2, k_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3} \tag{6}$$

where

$$\begin{aligned}
&\Theta^{(2)}(k_1, k_2, k_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3} \\
&\quad = \tfrac{1}{4}\gamma\kappa[k_{2(\alpha_1}k_{3\beta_1)}(2\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2} - \eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}) \\
&\qquad + 4k_{3(\alpha_2}\eta_{\beta_2)(\alpha_1}\eta_{\beta_1)(\alpha_3}k_{2\beta_3)} + k_2\cdot k_3(\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1}\eta_{\alpha_3\beta_3} + \eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1}\eta_{\alpha_2\beta_2} \\
&\qquad - 2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)(\alpha_3}\eta_{\beta_3)\beta_1} - 2\eta_{\alpha_1(\alpha_3}\eta_{\beta_3)(\alpha_2}\eta_{\beta_2)\beta_1})] + \text{cyclic }(1, 2, 3)
\end{aligned}$$

$$\begin{aligned}
&\Theta^{(4)}(k_1, k_2, k_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3} \\
&\quad = -\tfrac{1}{2}\beta\kappa^3[2k_1^2\eta_{\alpha\beta_1}k_{3(\alpha_2}\eta_{\beta_2)(\alpha_3}k_{2\beta_3)} + (k_2^2 + k_3^2) \\
&\qquad \times (k_{2(\alpha_1}k_{3\beta_1)}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3} + \tfrac{1}{2}k_1^2\eta_{\alpha_1\beta_1}\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2}) + \tfrac{1}{2}k_1^4\eta_{\alpha_1\beta_1} \\
&\qquad \times (\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2} + \tfrac{1}{2}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3})] + \tfrac{1}{4}\alpha\kappa^3[4(k_1^2 + k_2^2 + k_3^2) \\
&\qquad \times (k_{3(\alpha_2}\eta_{\beta_2)(\alpha_1}\eta_{\beta_1)(\alpha_3}k_{2\beta_3)} + \tfrac{1}{2}k_{2(\alpha_1}k_{3\beta_1)}\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2})
\end{aligned}$$

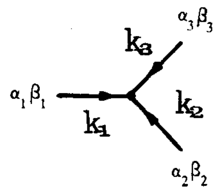

Fig. 1 The three-graviton vertex.

$$+ 2k_1^2\eta_{\alpha_1\beta_1}k_{3(\alpha_2}\eta_{\beta_2)(\alpha_3}k_{2\beta_2)} - k_1^2k_{2(\alpha_1}k_{3\beta_1)}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}$$

$$+ 2k_2^2k_3^2\eta_{\alpha_1\beta_1}\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2} - k_1^2k_2 \cdot k_3\eta_{\alpha_1\beta_1}(\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2} - \tfrac{1}{2}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3})$$

$$+ k_1^4(\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)(\alpha_3}\eta_{\beta_3)\beta_1} + \eta_{\beta_1(\alpha_2}\eta_{\beta_2)(\alpha_3}\eta_{\beta_3)\alpha_1})] + \text{cyclic } (1, 2, 3)$$

where $\Theta^{(2)}$ is the contribution of the term R in the action (Capper *et al.*, 1973) and $\Theta^{(4)}$ is contributed by terms R^2 and $R_{\mu\nu}R^{\mu\nu}$.

This result shows that the three-graviton vertex behaves as k^4 at large momentum. By a similar calculation, one can predict that the ultraviolet behaviors of four, five, . . .-graviton vertices are also $\sim k^4$. In some references the three-graviton vertex behavior was considered only as k^2 (Fig. 1). In this paper we believe that the higher-order terms of momentum in the graviton vertex may be neglected when discussing its infrared behavior, but when discussing its ultraviolet behavior the terms like k^4 will play a major role.

2. DIVERGENCE ANALYSIS OF ONE-LOOP DIAGRAMS

In the following we shall discuss the divergence for various one-loop diagrams in the higher-derivative gravity.

2.1. The Case with Purely 2-Derivative Graviton Vertex

If the graviton vertex is only counted to 2-derivative order, its ultraviolet behavior is like k^2. Divergent diagrams with pure external graviton lines are illustrated in Fig. 2. It is shown that there is an infinite number of divergent

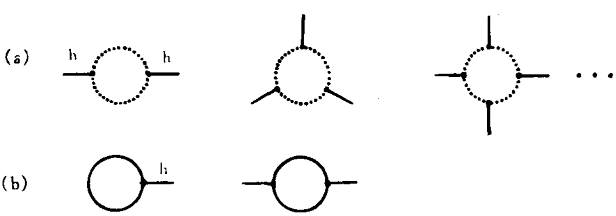

Fig. 2 The two types of divergent diagram which involve pure external graviton lines with purely 2-derivative graviton vertices. (a) Ghost loop, (b) graviton loop.

ghost-loop diagrams, and only two of the graviton-loop diagrams, which involve one- or two-graviton vertices, are divergent. By analyzing the degree of divergence, each of the ghost-loop diagrams in Fig. 2(a) is fourth-order divergent, and the graviton-loop diagrams in Fig. 2(b) are quadratically and logarithmically divergent, respectively. But the other graviton-loop diagrams, involving three and more graviton vertices, will converge.

For the diagrams involving external ghosts, there are three types of divergent diagram, illustrated in Fig. 3. Each of these three types has an infinite number of divergent diagrams, and all are linearly divergent. But those ghost diagrams in which graviton vertices are included will converge.

As is well known, all the divergences illustrated in Fig. 2 and 3 can be removed by nonlinear field renormalization and renormalization of the coupling constants for the case with a purely 2-derivative graviton vertex.

2.2. The Case with 4-Derivative Graviton Vertex

In the higher-derivative quantum gravity, the gravitational action includes quadratic terms of curvature tensors besides curvature scalars, and the contribution from the higher-derivative terms is included in the graviton propagator. There is no reason that this contribution (from the terms like R^2) could not be included in graviton vertices. So the three-graviton vertex should be given by (6), i.e., its ultraviolet behavior will be like k^4 not like k^2. Thus the divergence of the higher-derivative quantum gravity should be discussed further.

If we consider the 4-derivative contribution to the graviton vertex, the divergence illustrated in Figs. 2(a) and 3 is the same as the above, but both of those in Fig. 2(b) are now fourth-order divergent. The theory also contains the following types of new divergence: (1) graviton-loop diagrams with three and more graviton vertices, illustrated in Fig. 4, and (2) three types of diagram involving external ghosts which involve one or more graviton vertices, as in Fig. 5.

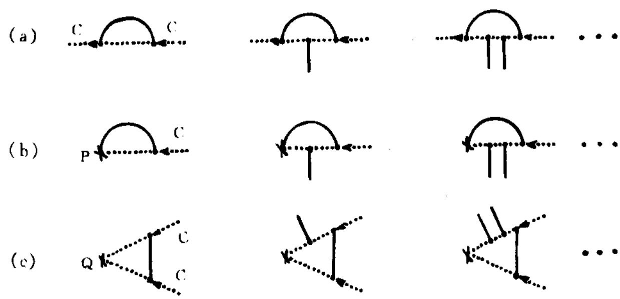

Fig. 3 The three types of divergent diagram involving external ghost lines with purely 2-derivative vertices. (a) CC type, (b) P type, (c) Q type.

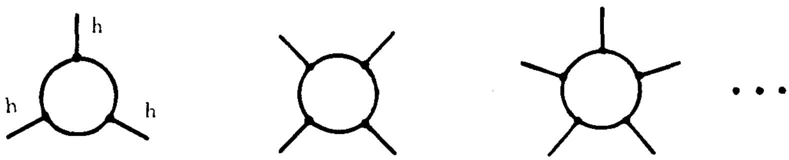

Fig. 4 Divergent graviton-loop diagrams involving three or more graviton lines with 4-derivative graviton vertices.

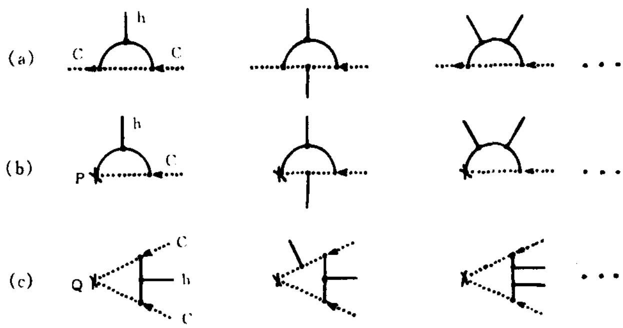

Fig. 5 Divergent diagrams involving external ghosts with the 4-derivative vertices. (a) $\overline{C}C$ type, (b) P type, (c) Q type.

Each of the diagrams in Fig. 4 is fourth-order divergent. Its degree of divergence is independent of the number of graviton vertices and the number of external graviton lines involved in the diagrams. So these types of diagrams are infinite.

For the three types of divergent diagrams involving external ghosts in Fig. 5, the degree of divergence of each is 1 and is independent of the number of graviton vertices included in these diagrams. So each of these three types has an infinite number of divergences also.

3. DIVERGENCE ANALYSIS OF ARBITRARY LOOP DIAGRAMS

In general, for an arbitrary 1PI diagram let I_c be the number of internal ghost lines, E_c the number of external ghost lines, $E_{\bar{c}}$ the number of external antighost lines, n_P the number of P-type vertices, n_Q the number of Q-type vertices, n_c the number of h-C-$\overline{C}$ vertices, and n_h the number of graviton vertices. By considering the results in Section 1 and noticing that each external antighost line carries with it a factor of external momentum, the degree of an arbitrary diagram is

$$D = 4 - 2n_P - n_Q - E_c - 2E_{\bar{c}} \tag{7}$$

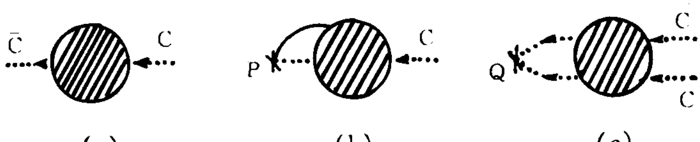

Fig. 6 The three types of divergent diagram which involve external ghost lines. (a) $\bar{C}C$ type, (b) P type, (c) Q type.

where we have used the topological relation

$$2(I_c - n_c) = 2n_Q - n_P - E_c - E_{\bar{c}}$$

and considered the three-graviton vertex behavior as k^4 in the ultraviolet region.

Together with the conservation of ghost number, equation (7) enables us to catalog three different types of divergent structures involving external ghosts. These are illustrated in Fig. 6. In the theory only counting 2-derivative graviton vertices, each of the three types has degree of divergence $D = 1 - 2n_h$, i.e., the divergent diagrams which involve external ghosts are only those without a graviton vertex, and the others, with one and more graviton vertices, do not diverge. But when the 4-derivative contribution is included in the graviton vertex, all those illustrated in Fig. 6 have degree of divergence $D = 1$, and the degree is independent of the number of included graviton vertices. So those ghost diagrams which include graviton vertices and do not diverge in the former theory do diverge now, and all are linearly divergent. So, of course, we have infinite such divergences.

In the theory only counting the 2-derivative graviton vertex, all the diagrams whose external lines are all gravitons, as in Fig. 7, have degree of divergence $D = 4 - 2n_h$, i.e., their degree depends on the number of graviton vertices. Consequently, all the divergences which involve pure external graviton lines only have $n_h \leqslant 2$. And other diagrams which involve three and more graviton vertices will converge (for $D < 0$). But counting the 4-derivative contribution to the graviton vertex, the diagrams with pure external graviton lines are all fourth-order divergent and their degree is independent, from (7), of the number of graviton vertices. So those diagrams which involve three and more graviton vertices also diverge now, and are all fourth-order divergent. The number of such divergences, of course, is infinite also.

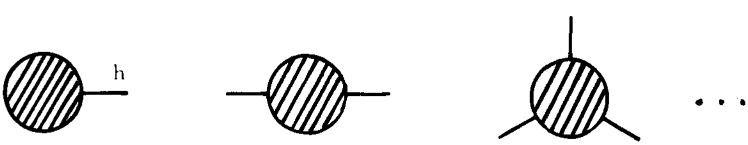

Fig. 7 The divergent diagrams with pure external graviton lines.

From the analysis above, we have shown that an infinite number of new divergences will emerge in the higher-derivative quantum gravity when the 4-derivative contribution is included in the graviton vertex.

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